Int J Theor Phys (2009) 48: 3220–3227
DOI 10.1007/s10773-009-0123-x

Generalized Uncertainty Relation and Hawking Radiation of the Black Hole

Ren Zhao · Lichun Zhang · Yueqin Wu · Huaifan Li

Received: 7 June 2009 / Accepted: 3 August 2009 / Published online: 19 August 2009

Abstract Recently, there has been much attention devoted to the correction to the black hole radiation spectrum and the quantum corrections to Bekenstein-Hawking entropy. In particular, many researchers have expressed a vested interest in the coefficient of the logarithmic term of the black hole entropy correction term. In this paper, we calculate the radiation spectrum of arbitrary dimension Schwarzschild black hole after considering the generalized uncertainty principle. The correction value of Bekenstein-Hawking entropy is derived.

Keywords Generalized uncertainty principle · Bekenstein-Hawking entropy · Tunnel radiation

1 Introduction

Hawking [1] interpreted the quantum effect of the black hole as emitting thermal radiant spectrum from event horizon. The discovery of this effect not only solved the problem in black hole thermodynamics but also announced the relation among quantum mechanics, thermodynamics and gravitation. Discovering the thermal properties of various black holes is an important subject of black hole physics. Hawking pointed that vacuum fluctuations near the surface of the black hole would produce virtual particle pair. When the virtual particles with negative energy come into black hole via tunnel effect, the energy of the black hole will decrease. At the same time, the particle with positive energy may thread out the gravitation

This research is supported by the Natural Science Foundation of Shanxi Province, China (Grant No. 2006011012) and the Shanxi Datong University doctoral Sustentation Fund, China.

R. Zhao (✉) · L. Zhang · Y. Wu · H. Li
Institute of Theoretical Physics, Shanxi Datong University, Datong 037009, China
e-mail: Zhaoren2969@yahoo.com.cn

R. Zhao · L. Zhang · Y. Wu
Department of Physics, Shanxi Datong University, Datong 037009, China

H. Li
Department of Applied Physics, Xi'an Jiaotong University, Xi'an 710049, China

region outside the black hole. Equivalently, the black hole radiates a particle. Gibbons and Hawking [2] proven that the radiation spectrum of the black hole is a strict black body spectrum. This discovery is of positive significance for understanding and investigating star evolution,which greatly promotes the development of the black hole thermodynamics. In the past decades, based on the research on the black hole radiation and the property of the black hole thermodynamics, the four laws of the black hole thermodynamics are derived [3, 4]. And the thermal radiation of various black holes has been studied [5–13]. They all derived that the radiation spectrum of the black hole is an exact black body spectrum. There are two obvious disputes: the first is information loss. Hawking radiation makes the people realize that the black hole is no longer an ultimate state of the star. The black hole can evolve and finally disappear. Since Hawking radiation parameter from black body spectrum. Thus the black hole radiation will not take any information of an inner matter of the black hole. This means when the black hole has vaporized all information including unitary property will lose. The information loss of the black hole means the pure quantum state will decay to mixed state. This violates the unitary principle in quantum mechanics. This challenges the foundation theory of quantum mechanics. The second is the reaction of radiation to the spacetime. Because the black hole produces Hawking radiation, state parameters(energy and charge) describing the black hole will fluctuate. However, in the process of proving Hawking radiation, the affect of radiation on the black hole state parameters was not considered in the past. That is, the reaction of radiation to spacetime is not considered. So only under the condition that the spacetime is invariable, Hawking radiation derives the purely thermal spectrum.

In 2000, Parikh and Wilczek put forward a semiclassical method for calculating the correction spectrum of the black hole Hawking radiation [14]. In this method, the black hole Hawking radiation is understood as a sort of quantum tunneling. The potential barrier is determined by the energy of emission particles. The key of this method is in emphasizing energy conservation during the particle emission process and establishing a good coordinate system at horizon. Using this method Parikh and Wilczek have calculated the radiation spectrum of particles when it tunnels horizons of a Schwarzschild and a Reissner-Nordström black hole. The result departs from the purely thermal spectrum. It satisfies unitary principle and information conservation. Subsequently, the Hawking radiation correction spectrums of various black holes have been calculated [15–29]. The outgoing rate of the black hole radiation particle is

$$\Gamma = \exp[\Delta S], \tag{1}$$

where ΔS is the change of Bekenstein-Hawking (B-H) entropy before and after the black hole radiation. The results all satisfy unitary theory and support conservation of information in Hawking radiation. Recently, Refs. [30, 31] starts from the Damour-Ruffini's method [5], and also derived that the outgoing rate of the black hole radiation particle is (1).

The correction to B-H entropy of black hole attracts researcher's attention. Various method has been used to discuss the correction value to black hole B-H entropy [32–45]. Many researchers think that the expression of the correction to B-H entropy of the black hole is

$$S = \frac{A}{4G} + \chi \ln \frac{A}{4G}, \tag{2}$$

where A is the area of the black hole horizon, χ is a dimensionless constant. However, the exact coefficient of logarithmic term is not derived. Some reference gives $\chi = -3/2$, for example, [34]; some works lead to $\chi = -1/2$, for example, [46–48]; the paper [49] argued that χ should be a positive integer and on the other hand, the author of [50] argued that χ should be equal to zero.

Because Hawking radiation of the black hole is a quantum effect of the black hole, the radiation particle or absorption particle must consider the uncertainty principle. However, Heisenberg uncertainty principle does not hold when gravity action exist. And the generalized uncertainty principle replaces it. Many attention have been paid to the correction to radiation spectrum of the black hole due to generalized uncertainty principle [40–43]. In this paper, considering the generalized uncertainty principle and applying to tunneling method we discuss radiation spectrum of the black hole. The radiation spectrum and the correction value of Bekenstein-Hawking entropy are derived. Throughout the study, the units $G = c = \hbar = 1$.

2 Tunneling Property of Particle at the Black Hole Horizon

The metric of an arbitrary dimensional Schwarzschild black hole could be written [43]

$$ds^2 = -f(r)dt_s^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{n+1}^2, \tag{3}$$

where

$$f(r) = 1 - \frac{r_0^n}{r^n}, \quad r_0^n = \frac{16\pi M}{(n+1)\Omega_{n+1}}, \tag{4}$$

M is the ADM mass of the black hole and Ω_{n+1} is the metric of the unit S^{n+1}. The event horizon is located at $r_h = r_0$. The B-H entropy of the black hole is

$$S_{BH} = \frac{1}{4} A_h = \frac{1}{4}\Omega_{n+1} r_0^{n+1}. \tag{5}$$

The coordinate singularity at the horizon is removed by going to Painleve coordinates [51]. Under the transformation

$$dt_s = dt - \frac{\sqrt{1 - f(r)}}{f(r)} dr, \tag{6}$$

the metric (3) takes the form

$$ds^2 = -f(r)dt^2 + 2\sqrt{1 - f(r)}dtdr + dr^2 + r^2 d\Omega_{n+1}. \tag{7}$$

The basic idea of this method is to find the radial null geodesics for the metric (7):

$$\dot{r} \equiv \frac{dr}{dt} = \pm 1 - \sqrt{1 - f(r)}, \tag{8}$$

where $+(-)$ sign gives outgoing (ingoing) null radial geodesics. Using this one has to calculate the imaginary part of the action for a shell with energy ω. In the original work [14], the imaginary part of the action is defined as,

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} p_r dr = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r' dr = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{H} \frac{dH'}{\dot{r}} dr, \tag{9}$$

where in the last step we multiply and divide the integrand by the two sides of Hamilton's equation $\dot{r} = \frac{dH}{dp_r}|_r$.

Near the horizon one can expand $f(r)$ about the horizon r_h.

$$f(r) = f'(r_h)(r - r_h) + o((r - r_h)^2). \tag{10}$$

Substituting these in (8) $\dot{r}$ can be approximately expressed as

$$\dot{r} \approx \frac{1}{2} f'(r_h)(r - r_h). \tag{11}$$

The surface gravity of black hole on the horizon is defined by

$$\kappa(M) = \Gamma^0_{00}\Big|_{r=r_h} = \frac{1}{2}\left[\sqrt{1 - f(r)}\frac{df(r)}{dr}\right]_{r=r_h}. \tag{12}$$

From (12) and (11), we have

$$\dot{r} \approx 2\pi T(M)(r - r_h). \tag{13}$$

Now taking into account the self-gravitation effects [52], the above integration can be expressed as

$$\operatorname{Im} S = \operatorname{Im} \int_{r_{in}}^{r_{out}} \int_{M}^{M-\omega} \frac{dH'}{\dot{r}} dr = -\operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{d\omega'}{\dot{r}} dr, \tag{14}$$

where we used the fact that the Hamiltonian $H = M - \omega$, with M being the original mass of the black hole. Here $\dot{r}$ can be approximated by (13) as follows

$$\dot{r} \approx 2\pi(r - r_h)T(M - \omega), \tag{15}$$

where r_h is the modified arbitrary dimensional Schwarzschild black hole radius and $T(M - \omega)$ is the modified horizon temperature. Substituting (15) into (14), we get

$$\operatorname{Im} S = -\operatorname{Im} \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{d\omega'}{2\pi(r - r_h)T(M - \omega')} dr. \tag{16}$$

But now the integration over r can be done by deforming the contour. Ensuring that the positive energy solutions decay in time we have after r integration [45]

$$\operatorname{Im} S = \pi \int_0^{\omega} \frac{d\omega'}{2\pi T(M - \omega')}. \tag{17}$$

After considering generalized uncertainty principle, the temperature of the black hole is [43]

$$T(M) = \frac{n\lambda}{2\pi\alpha^2} M^{1/n}\left[1 - \sqrt{1 - \frac{\alpha^2}{\lambda^2 M^{2/n}}}\right], \tag{18}$$

where $\lambda = (\frac{16\pi}{(n+1)\Omega_{n+1}})^{1/n}$ and α is a dimensionless constant of order one. Equation (18) may be Taylor expanded around $\alpha = 0$

$$T(M) = \frac{n}{4\pi\lambda} M^{-1/n}\left[1 + \frac{\alpha^2}{4\lambda^2 M^{2/n}} + \cdots\right]. \tag{19}$$

Substituting (19) into (17)

$$\begin{aligned}\operatorname{Im} S &= 2\pi \int_0^{\omega} \left[\frac{\lambda}{n}(M-\omega')^{1/n}\left(1 - \frac{\alpha^2}{\alpha^2 + 4\lambda^2(M-\omega')^{2/n}(1+\cdots)} - \cdots\right)\right] d\omega' \\ &\approx 2\pi \int_0^{\omega} \left[\frac{\lambda}{n}(M-\omega')^{1/n}\left(1 - \frac{\alpha^2}{\alpha^2 + 4\lambda^2(M-\omega')^{2/n}} - \cdots\right)\right] d\omega'. \end{aligned} \tag{20}$$

For four-dimensional Schwarzschild black hole $n = 1, \lambda = 2$, we integrate (20) and obtain

$$\operatorname{Im} S = 4\pi\omega\left(M - \frac{\omega}{2}\right) - \frac{\pi}{8}\alpha^2 \ln\left(\frac{\alpha^2 + 16M^2}{\alpha^2 + 16(M-\omega)^2}\right). \tag{21}$$

Now according to the WKB-approximation method the tunneling probability is given by

$$\Gamma \sim e^{-2 \operatorname{Im} S}. \tag{22}$$

From (21), considering the generalized uncertainty principle the probability that particle through potential barrier is

$$\Gamma \sim \left[\frac{\alpha^2 + 16M^2}{\alpha^2 + 16(M-\omega)^2}\right]^{\alpha^2\pi/4} e^{-8\pi\omega(M-\frac{\omega}{2})}. \tag{23}$$

3 Correction to B-H Entropy after Considering the Generalized Uncertainty Principle

It is known [14–17] that the change in the B-H entropy due to the tunneling through the horizon is related to $\operatorname{Im} S$ by the following relation

$$\Delta S_{bh} = -2 \operatorname{Im} S. \tag{24}$$

Therefore the corrected change in B-H entropy is

$$\Delta S_{bh} = -8\pi\omega\left(M - \frac{\omega}{2}\right) + \frac{\pi}{4}\alpha^2 \ln\{\alpha^2 + 16M^2\} - \frac{\pi}{4}\alpha^2 \ln\{\alpha^2 + 16(M-\omega)^2\}. \tag{25}$$

(25) is the change of B-H entropy after considering generalized uncertainty principle. That is

$$S_{final}(M-\omega) - S_{initial}(M) = \Delta S_{bh}. \tag{26}$$

From (26), when $M = \omega$, $-S_{intial} = \Delta S_{bh}|_{M=\omega}$. So the B-H entropy of the black hole with mass M is

$$\begin{aligned} S_{bh} &= 4\pi M^2 - \frac{\pi}{4}\alpha^2 \ln\left(1 + \frac{16M^2}{\alpha^2}\right) \\ &= S_{BH} - \frac{\pi}{4}\alpha^2 \ln S_{BH} - \frac{\pi^2\alpha^2}{16 S_{BH}} + \cdots. \end{aligned} \tag{27}$$

Comparing (27) with (2), we obtain that $\chi = -\frac{\pi}{4}\alpha^2$ is negative in (2). The result is correspond with that of Ref. [32]. Based on (27), the coefficients of logarithmic correction term in correction to B-H entropy depend on α introduced in the generalized uncertainty principle. If we can determine α, the coefficients of logarithmic correction term in correction to B-H entropy will be determined. When $n \neq 1$, we integrate (20)

$$\begin{aligned}
\operatorname{Im} S &\approx 2\pi \int_0^{\omega} \left[\frac{\lambda}{n}(M-\omega')^{1/n}\left(1-\frac{\alpha^2}{\alpha^2+4\lambda^2(M-\omega')^{2/n}}-\cdots\right)\right]d\omega' \\
&= \frac{2\pi\lambda}{n+1}\left(M^{\frac{n+1}{n}}-(M-\omega)^{\frac{n+1}{n}}\right) \\
&\quad - \frac{\pi\alpha^2}{2\lambda}\left[\frac{1}{n-1}\left(M^{\frac{n-1}{n}}-(M-\omega)^{\frac{n-1}{n}}\right)\right. \\
&\quad \left. - \frac{\alpha^2}{4\lambda^2}\frac{1}{n-3}\left(M^{\frac{n-3}{n}}-(M-\omega)^{\frac{n-3}{n}}\right)+\cdots\right]. \qquad (28)
\end{aligned}$$

Thus

$$\begin{aligned}
S_{bh} &= \frac{4\pi\lambda}{n+1}M^{\frac{n+1}{n}} - \frac{\pi\alpha^2}{\lambda}\left[\frac{1}{n-1}M^{\frac{n-1}{n}} - \frac{\alpha^2}{4\lambda^2}M^{\frac{n-3}{n}}+\cdots\right] \\
&= S_{BH} - \frac{\pi\alpha^2}{\lambda}\frac{1}{n-1}\left(\frac{n+1}{4\pi\lambda}S_{BH}\right)^{(n-1)/(n+1)} \\
&\quad + \frac{\pi\alpha^4}{4\lambda^3}\frac{1}{n-3}\left(\frac{n+1}{4\pi\lambda}S_{BH}\right)^{(n-3)/(n+1)} + \cdots. \qquad (29)
\end{aligned}$$

4 Conclusion

So we derive the probability that radiation particle through potential barrier in arbitrary dimension Schwarzschild black hole considering the generalized uncertainty principle. This probability not only is related to change of B-H entropy but also includes the correction term.

For four-dimensional space-time, after considering the generalized uncertainty principle the coefficients of logarithmic correction term in correction to B-H entropy of Schwarzschild black hole is negative. However, in higher-dimensional space-time, the correction to B-H entropy of Schwarzschild black hole does not include logarithmic term. To determine the coefficients of logarithmic correction term in correction to B-H entropy of Schwarzschild black hole, we need consider many factors, such as thermal fluctuation and reaction of radiation. In this paper, we only derive the correction to entropy of the black hole due to the generalized uncertainty principle. We provide a method for discussing correction to B-H entropy of the black hole considering every factor.

References

1. Hawking, S.W.: Particle creation by black holes. Commun. Math. Phys. **43**, 199 (1975)
2. Gibbons, G.W., Hawking, S.W.: Action integrals and partition functions in quantum gravity. Phys. Rev. D **15**, 2752 (1977)
3. Bardeen, J.M., Carter, B., Hawking, S.W.: The four laws of black hole mechanics. Commun. Math. Phys. **31**, 161 (1973)
4. Wald, R.M.: General Relativity. University of Chicago Press, Chicago (1984)

5. Damour, T., Ruffini, R.: Black-hole evaporation in the Klein-Sauter-Heisenberg-Euler formalism. Phys. Rev. D **14**, 332 (1976)
6. Unruh, W.G.: Notes on black-hole evaporation. Phys. Rev. D **14**, 870 (1976)
7. Hartle, J.B., Hawking, S.W.: Path-integral derivation of black-hole radiance. Phys. Rev. D **13**, 2188 (1976)
8. Sannan, S.: Heuristic derivation of the probability distributions of particles emitted by a black hole. Gen. Relativ. Gravit. **20**, 239 (1988)
9. Gibbons, G.W., Perry, M.J., Pope, C.N.: The first law of thermodynamics for Kerr-anti-de Sitter black holes. Class. Quantum Gravity **22**, 1503 (2005)
10. Cai, R.G., Kim, S.P., Wang, B.: Ricci flat black holes and Hawking-Page phase transition in Gauss-Bonnet gravity and dilaton gravity. Phys. Rev. D **76**, 024011 (2007)
11. Cai, R.G.: Cardy–Verlinde formula and thermodynamics of black holes in de Sitter spaces. Nucl. Phys. B **628**, 375 (2002)
12. Cai, R.G., Cao, L.M.: Thermodynamics of apparent horizon in brane world scenario. Nucl. Phys. B **785**, 135 (2007)
13. Dehghani, M.H.: Charged rotating black branes in anti-de Sitter Einstein-Gauss-Bonnet gravity. Phys. Rev. D **67**, 064017 (2003)
14. Parikh, M.K., Wilczek, F.: Hawking radiation as tunneling. Phys. Rev. Lett. **85**, 5042 (2000)
15. Vagenas, E.C.: Are extremal 2D black holes really frozen? Phys. Lett. B **503**, 399 (2001)
16. Vagenas, E.C.: Two-dimensional dilatonic black holes and Hawking irradiation. Mod. Phys. Lett. A **17**, 609 (2002)
17. Vagenas, E.C.: Semiclassical corrections to the Bekenstein-Hawking entropy of the BTZ black hole via self-gravitation. Phys. Lett. B **533**, 302 (2002)
18. Vagenas, E.C.: Generalization of the KKW analysis for black hole radiation. Phys. Lett. B **559**, 65 (2003)
19. Setare, M.R., Vagenas, E.C.: Self-gravitational corrections to the Cardy-Verlinde formula of the Achúcarro-Ortiz black hole. Phys. Lett. B **584**, 127 (2004)
20. Kerner, R., Mann, R.B.: Tunnelling, temperature, and Taub-NUT black holes. Phys. Rev. D **73**, 104010 (2006)
21. Kerner, R., Mann, R.B.: Tunnelling from Gödel black holes. Phys. Rev. D **75**, 084022 (2007)
22. Kerner, R., Mann, R.B.: Charged fermions tunnelling from Kerr-Newman black holes. Phys. Lett. B **665**, 277 (2008)
23. Cai, R.G., Cao, L.M., Hu, Y.P.: Hawking radiation of apparent horizon in a FRW universe. Class. Quantum Gravity **26**, 155018 (2009)
24. Zhang, J.Y., Fan, J.H.: Tunnelling effect of charged and magnetized particles from the Kerr-Newman-Kasuya black hole. Phys. Lett. B **648**, 133 (2007)
25. Wu, X., Gao, S.: Tunneling effect near a weakly isolated horizon. Phys. Rev. D **75**, 044027 (2007)
26. Arzano, M., Medved, A.J.M., Vagenas, E.C.: Hawking radiation as tunneling through the quantum horizon. J. High Energy Phys. **09**, 037 (2005)
27. Jiang, Q.Q., Wu, S.Q., Cai, X.: Hawking radiation from dilatonic black holes via anomalies. Phys. Rev. D **75**, 064029 (2007)
28. Li, R., Ren, J.R.: Dirac particles tunneling from BTZ black hole. Phys. Lett. B **661**, 370 (2008)
29. Peng, J.J., Wu, S.Q.: Covariant anomalies and Hawking radiation from charged rotating black strings in anti-de Sitter spacetimes. Phys. Lett. B **661**, 300 (2008)
30. He, X.K., B Liu, W.: Modified Hawking radiation in a BTZ black hole using Damour-Ruffini method. Phys. Lett. B **653**, 330 (2007)
31. Zhou, S.W., Liu, W.B.: Hawking radiation of charged Dirac particles from a Kerr-Newman black hole. Phys. Rev. D **77**, 104021 (2008)
32. Medved, A.J.M., Vagenas, E.C.: When conceptual worlds collide: The generalized uncertainty principle and the Bekenstein-Hawking entropy. Phys. Rev. D **70**, 124021 (2004)
33. Chatterjee, A., Majumdar, P.: Universal canonical black hole entropy. Phys. Rev. Lett. **92**, 141301 (2004)
34. Kaul, R.K., Majumdar, P.: Logarithmic correction to the Bekenstein-Hawking entropy. Phys. Rev. Lett. **84**, 5255 (2000)
35. Camellia, G.A., Arzano, M., Procaccini, A.: Severe constraints on the loop-quantum-gravity energy-momentum dispersion relation from the black-hole area-entropy law. Phys. Rev. D **70**, 107501 (2004)
36. Chatterjee, A., Majumdar, P.: Mass and charge fluctuations and black hole entropy. Phys. Rev. D **71**, 024003 (2005)
37. Myung, Y.S.: Logarithmic corrections to three-dimensional black holes and de Sitter spaces. Phys. Lett. B **579**, 205 (2004)
38. Akbar, M.M., Das, S.: Entropy corrections for Schwarzschild and Reissner-Nordström black holes. Class. Quantum Gravity **21**, 1383 (2004)

39. Das, S., Majumdar, P., Bhaduri, R.K.: General logarithmic corrections to black-hole entropy. Class. Quantum Gravity **19**, 2355 (2002)
40. Zhao, R., Zhang, S.L.: Canonical entropy of three-dimensional BTZ black hole. Phys. Lett. B **641**, 318 (2006)
41. Zhao, R., Zhang, S.L.: Generalized uncertainty principle and black hole entropy. Phys. Lett. B **641**, 208 (2006)
42. Medved, A.J.M., Vagenas, E.C.: On Hawking radiation as tunneling with logarithmic corrections. Mod. Phys. Lett. A **20**, 1723 (2005)
43. Setare, M.R.: Corrections to the Cardy-Verlinde formula from the generalized uncertainty principle. Phys. Rev. D **70**, 087501 (2004)
44. Banerjee, R., Majhi, B.R.: Quantum tunneling beyond semiclassical approximation. J. High Energy Phys. **06**, 095 (2008)
45. Banerjee, R., Majhi, B.R.: Quantum tunneling and back reaction. Phys. Lett. B **662**, 62 (2008)
46. Ghosh, A., Mitra, P.: Log correction to the black hole area law. Phys. Rev. D **71**, 027502 (2005)
47. Domagala, M., Lewandowski, J.: Black-hole entropy from quantum geometry. Class. Quantum Gravity **21**, 5233 (2004)
48. Meissner, K.A.: Black-hole entropy in loop quantum gravity. Class. Quantum Gravity **21**, 5245 (2004)
49. Hod, S.: High-order corrections to the entropy and area of quantum black holes. Class. Quantum Gravity **21**, L97 (2004)
50. Medved, A.J.M.: A comment on black hole entropy or does nature abhor a logarithm? Class. Quantum Gravity **22**, 133 (2005)
51. Painlevé, P.: La mecanique classique el la theorie de la relativite. C. R. Acad. Sci. (Paris) **173**, 677 (1921)
52. Parikh, M.K., Wilczek, F.: Self-interaction correction to black hole radiance. Nucl. Phys. B **433**, 403 (1995)